Etienne Forest
High Energy Accelerator Research Organization
Tsubuka, Ibaraki, Japan

The Graduate University for Advanced Studies
(Sokendai)
Hayama, Kanagawa, Japan

Plan of the presentations

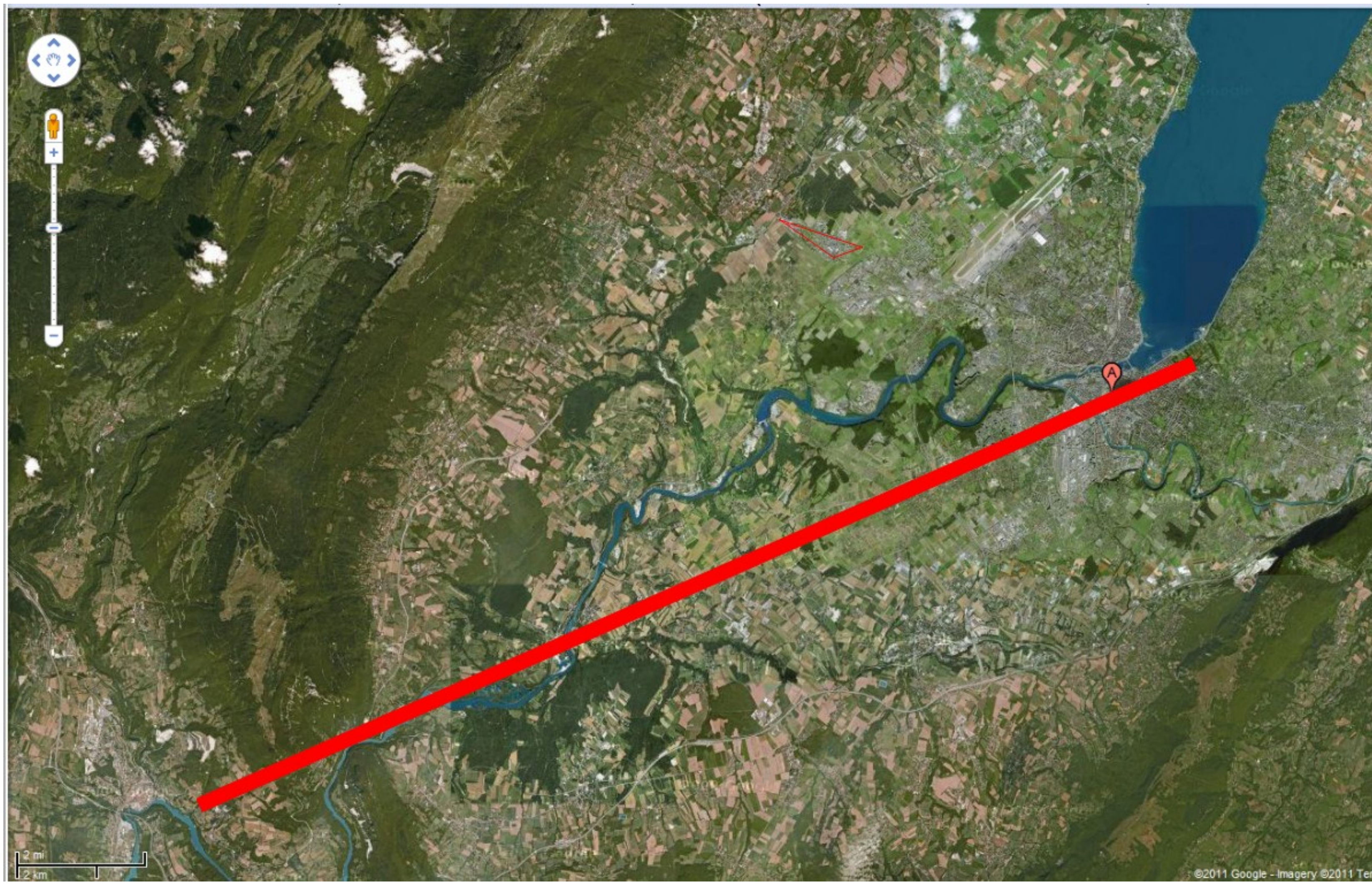
- Lens paradigm
- TPSA + a Small Code
- Geometric Integration: Almost indispensible in Rings
- Normal Form: "Universal Twiss Algorithm".
- Full exploitation of the lens paradigm: fibre structure (Not in SAD)

歴史的な脚注

The biggest accelerator, where is it? CERN はどこですか？

Interea ea legione quam secum habebat militibusque, その間に、（もともと）カエサルが持っていた一個軍団と
qui ex provincia convenerant, プローウィンキアから集まってきていた兵士達の手で
a lacu Lemanno, qui in flumen Rhodanum influit, ロダヌス河へと流れ込むレマンヌス湖から
ad montem Iuram, qui fines Sequanorum ab Helvetiis dividit, セークァニー族の領地とヘルウェティー族とを分けているユーラ山へと
milia passuum XVIIII murum in altitudinem pedum sedecim fossamque perducit. 十九マイルにわたって、高さ十六フィートの壁や壕をつくらせる。

IVLIVS GAIVS CAESAR, DE BELLO GALLICO, ~50BC (ユリウス ガイウス カエサル, ガリア戦記 ~50BC)



Salient Points 要点

1) Importance of single particle dynamics 動力学の重要性

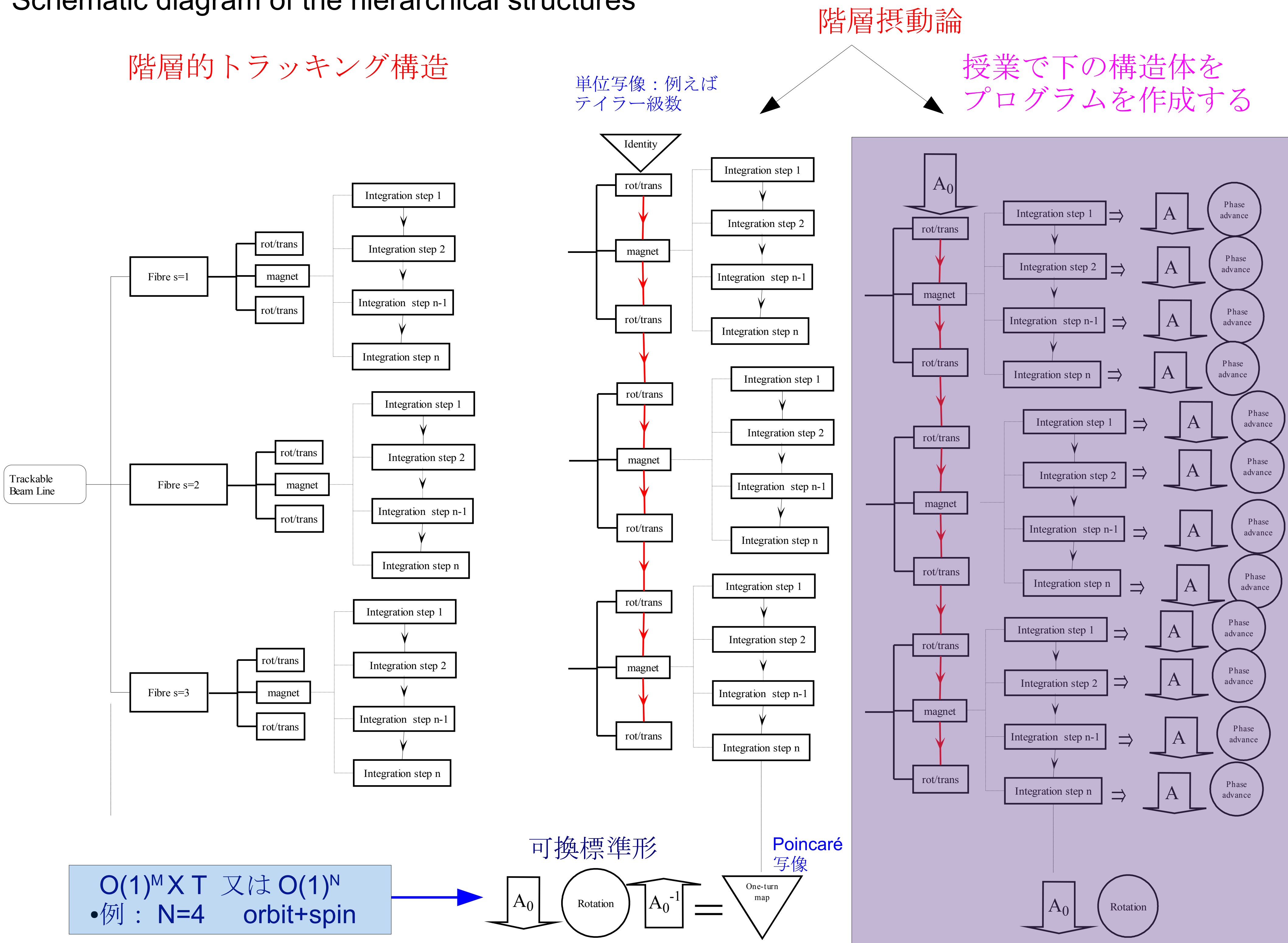
2)Lens based theory (technologically derived) レンズの定式化（技術から派生した）

3)Computers are hierarchical => search for a hierarchical theory 階層構造コンピューター＝＞階層構造理論 (layered structure)

4)Truncated Power Series Algebra and approximate Taylor Maps fits in that hierarchical theory(切り捨てられたテイラー級数の写像は、その階層理論に収まる）

5) The concept of a Normal Form is central to a hierarchical theory 標準形の概念は 階層理論の中心である

Schematic diagram of the hierarchical structures



A few points

Importance of single particle dynamics in accelerators as the starting point of more complex effects

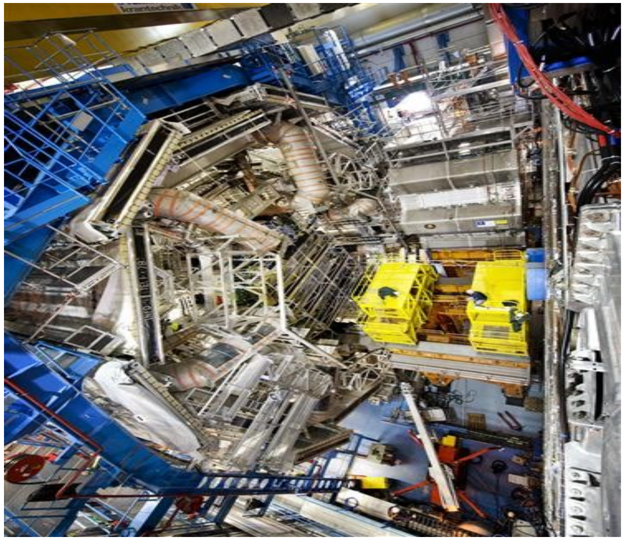
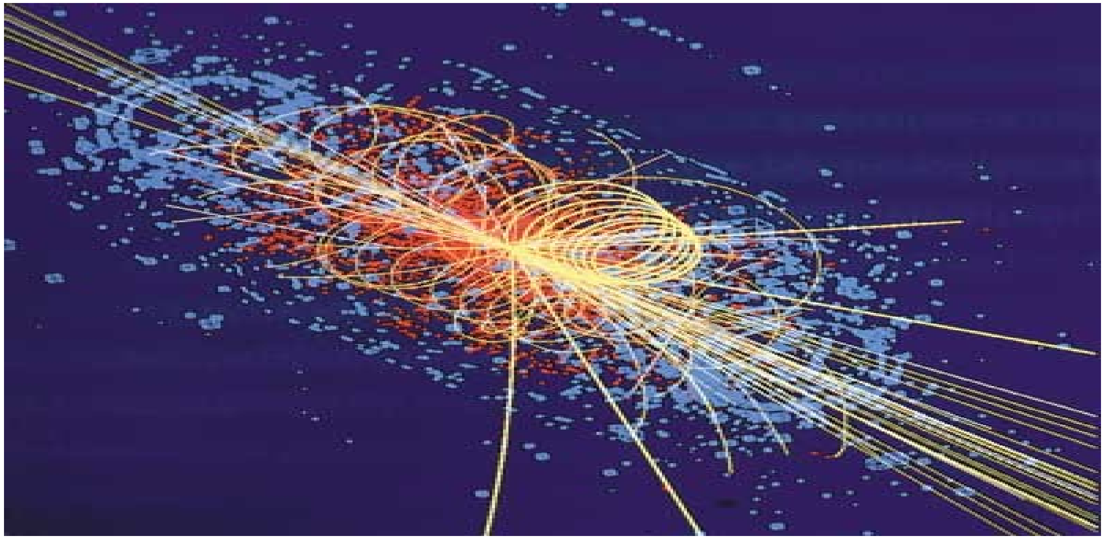
1) We first start a machine with low current: single particle dynamics holds perfectly. 低ビーム電流で始まる。

2) We then slowly built up the current. Due to the highly relavistic aspect of the machine, most collective effects are happening in a plane perpendicular to the motion. (Pancake model) β は 1 に近いので、空間電荷効果が軌道に直交です。

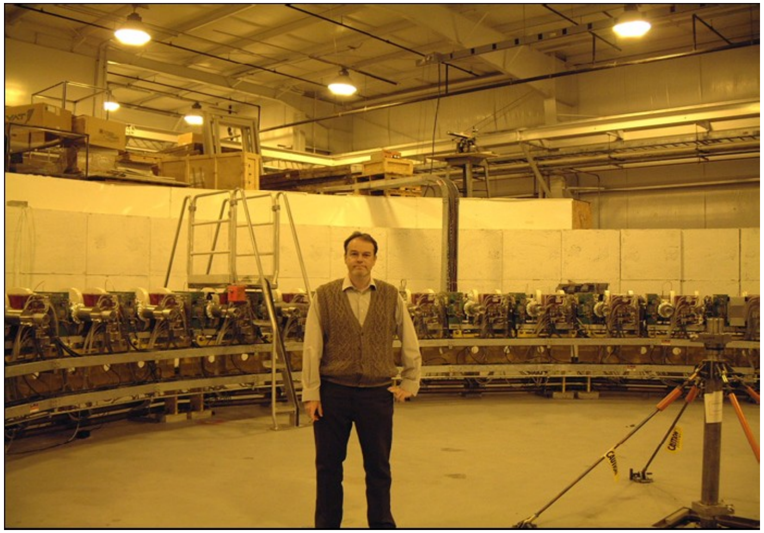
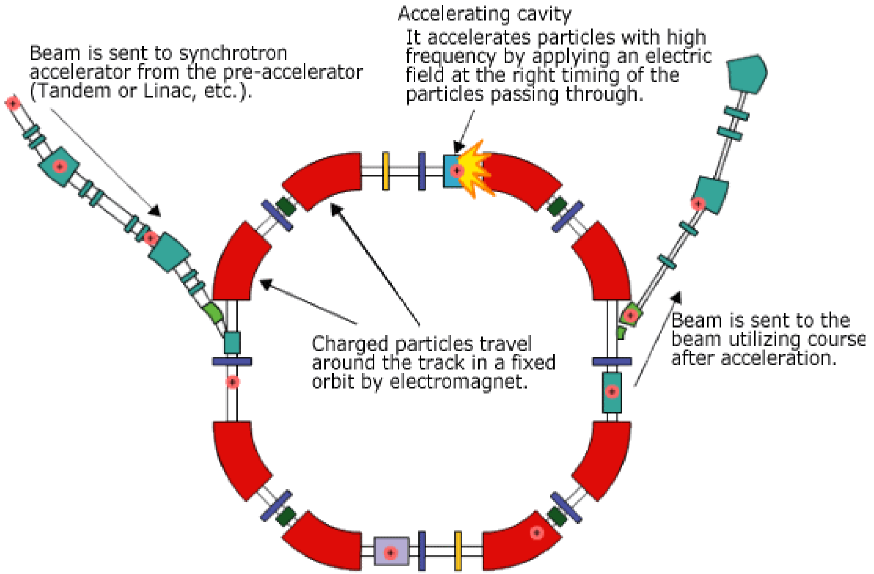
3) Therefore the lens paradigm I will describe, while 99% correct in the absence of collective effects, still holds well in the presence of collective phenomena. だからレンズの定式化 は、ほとんど有効です。

Constrasting system- システムの比較

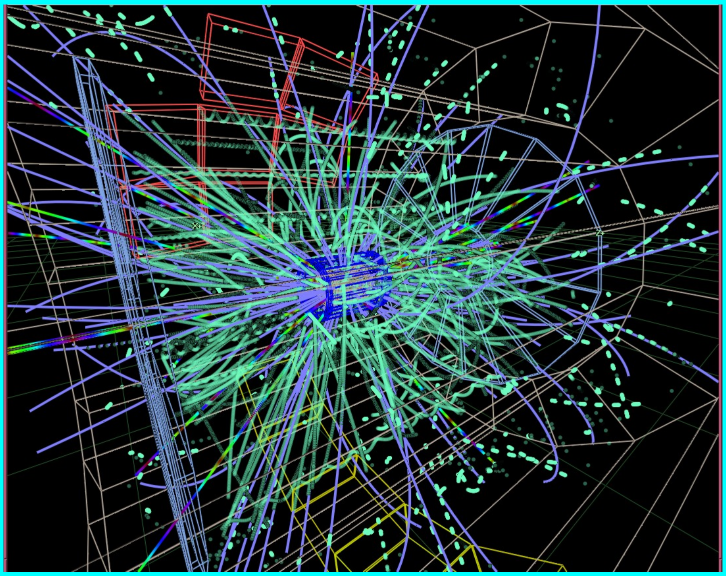
- Type 1) Detectors: F=ma（通常の物理学）

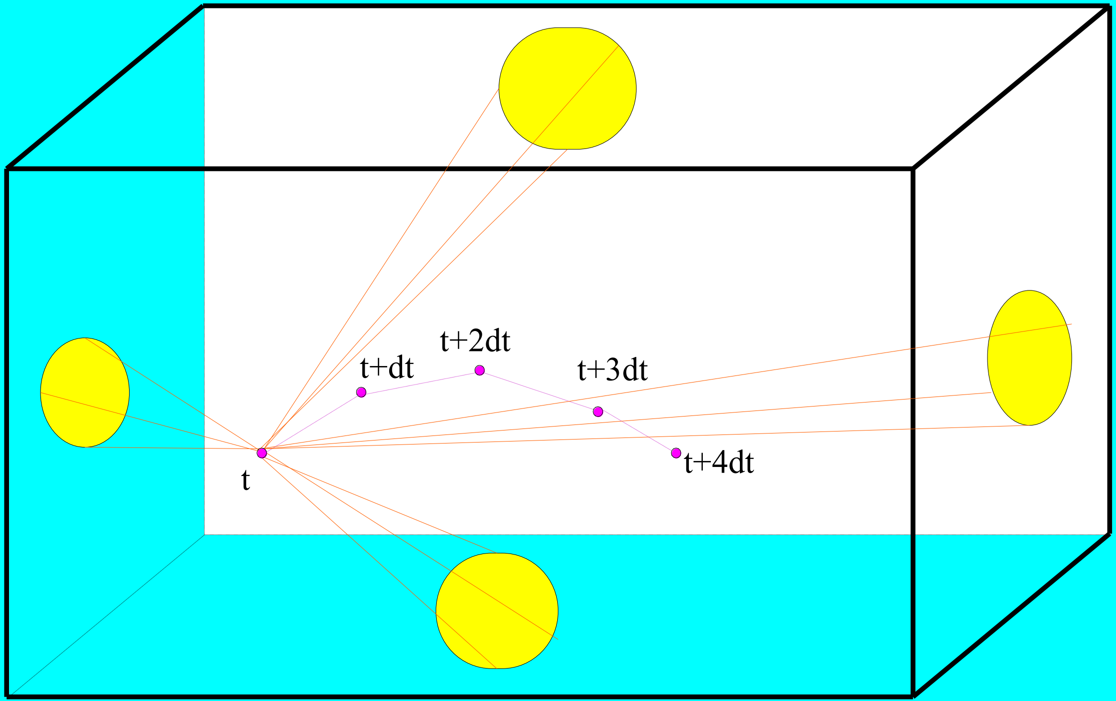
- Rings: Reformulate using **Lenses** レンズの観点から再定式化

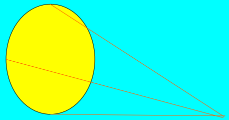



Ordinary: Detector, Planetary motion, etc..
普通のシステムのシステ：粒子検出器 , 惑星の動きなど ..



Real simulation





The yellow areas schematically represent coils or metal which produces the B-field in the device.The orange lines represent symbolically the "action at a distance."

Standard Strategy from High School Physics
標準的な方法（高校の物理）

1) Compute the magnetic field $\vec{B}$ at the position **x** of the particle in the device

2) The motion obeys Newton/Lorentz's laws

rest mass → m ; charge → q

$$\frac{d\vec{x}}{dt} = \vec{v}$$

$$\frac{d}{dt} m\gamma\vec{v} = q\,\vec{v} \times \vec{B}$$

3) Integrate numerically from time t to time t+dt

First Order Euler →

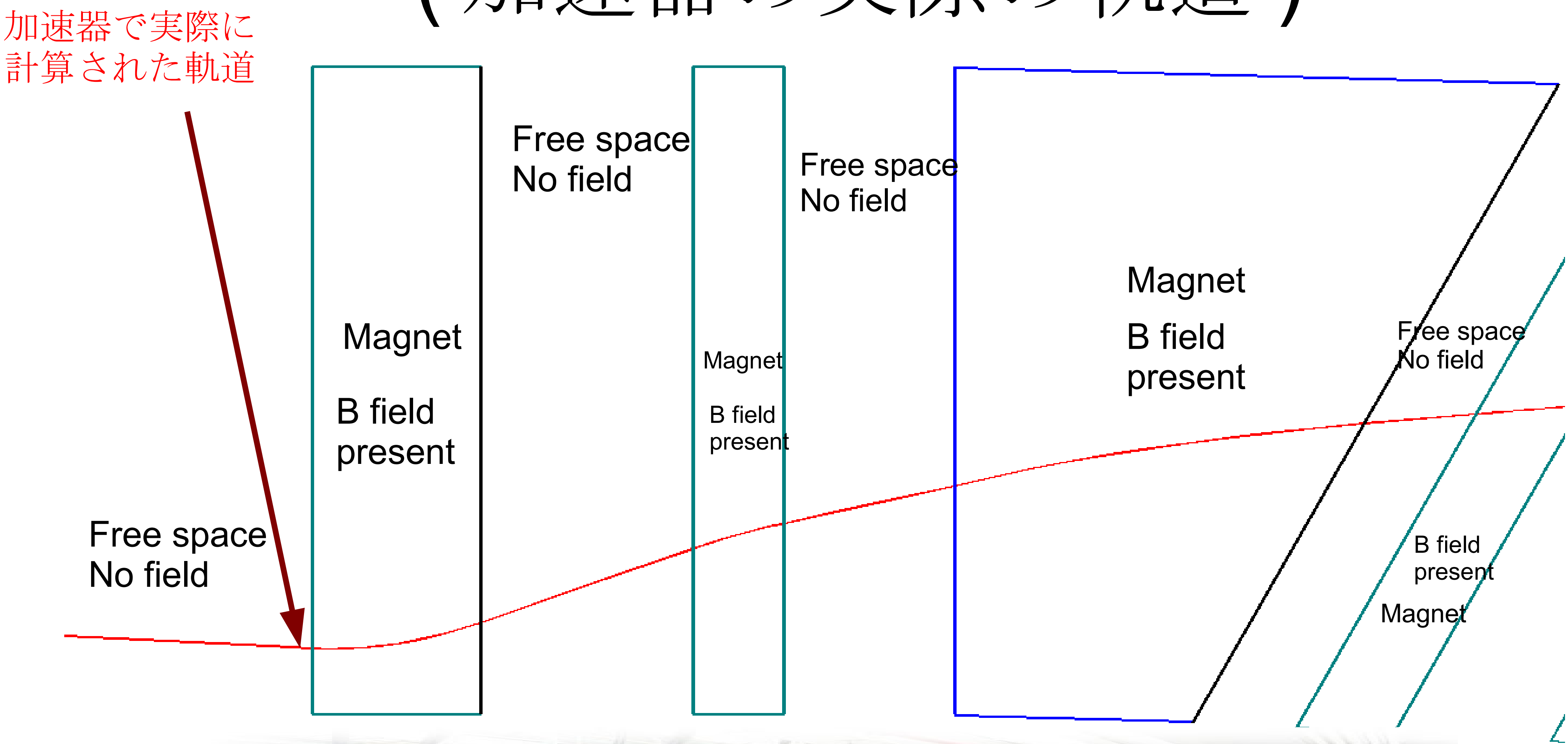
$$\vec{x}_{t+dt} \approx \vec{x}_t + dt\ \vec{v}_t$$

$$\vec{v}_{t+dt} \approx \vec{v}_t + dt\ \left\{ \frac{q}{m\gamma}\ \vec{v}_t \times \vec{B}(\vec{x}_t) \right\}$$

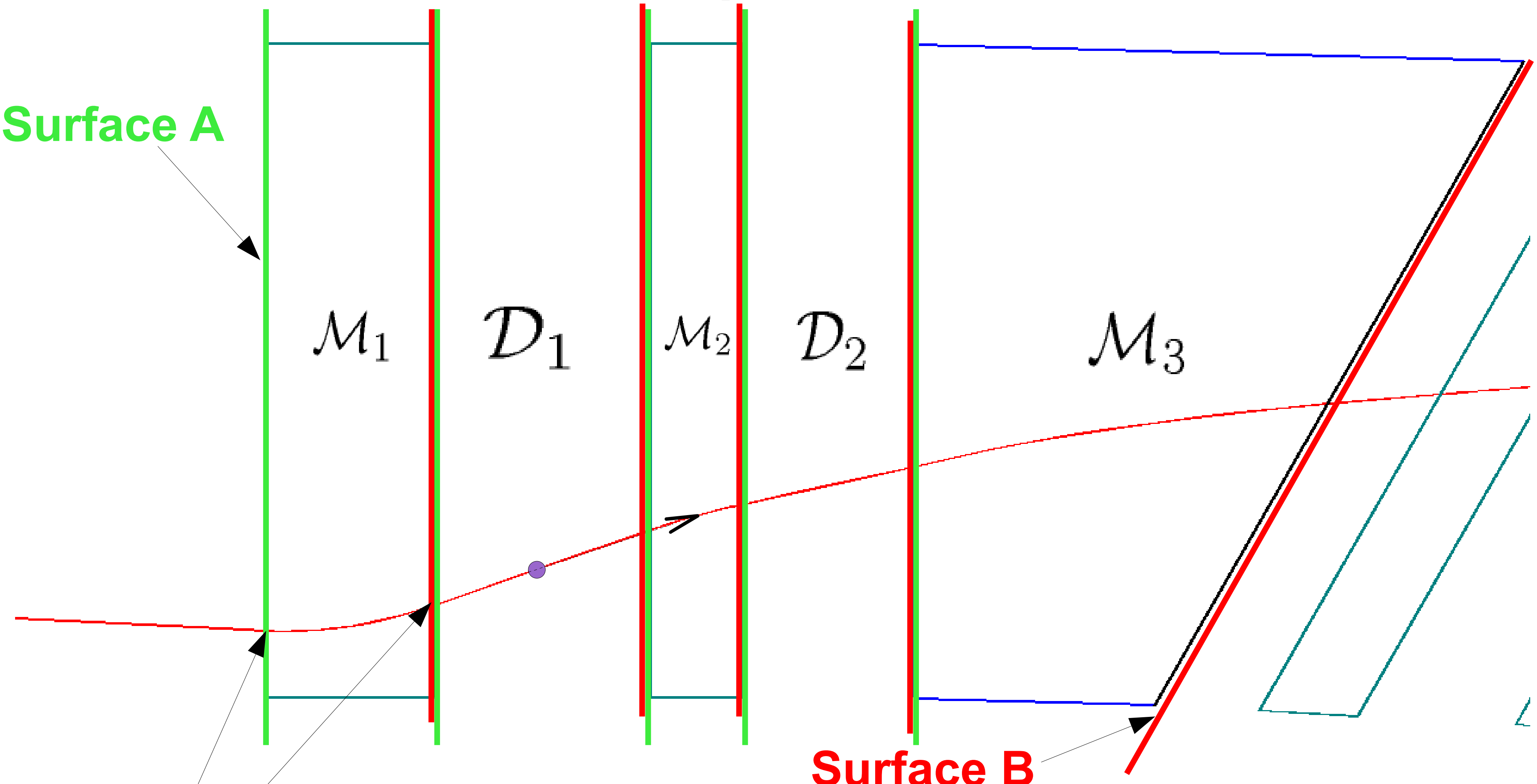
Now we look at an Accelerator

The Lens Formulation
レンズの定式化

Trajectory in an Accelerator
(加速器の実際の軌道)



Trivial Consequence 些細な結果

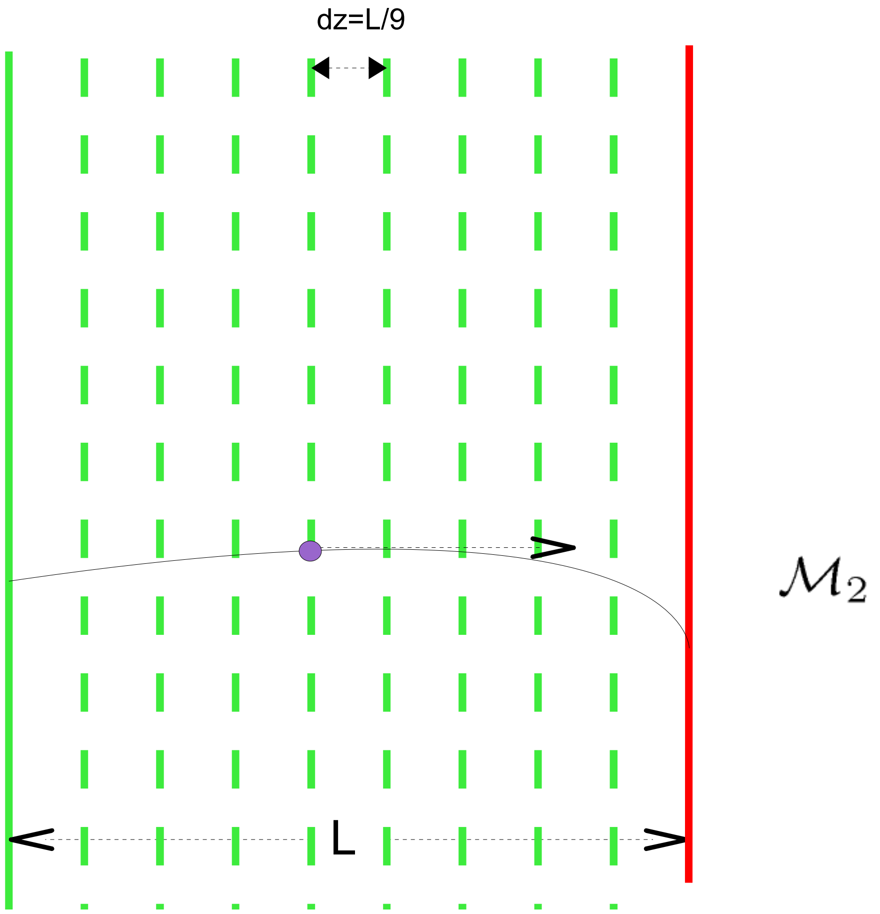


表面から写像を
使用することが出来る。

It is possible to concatenate (composed) maps into one big map.
レンズの写像の合成は出来る。

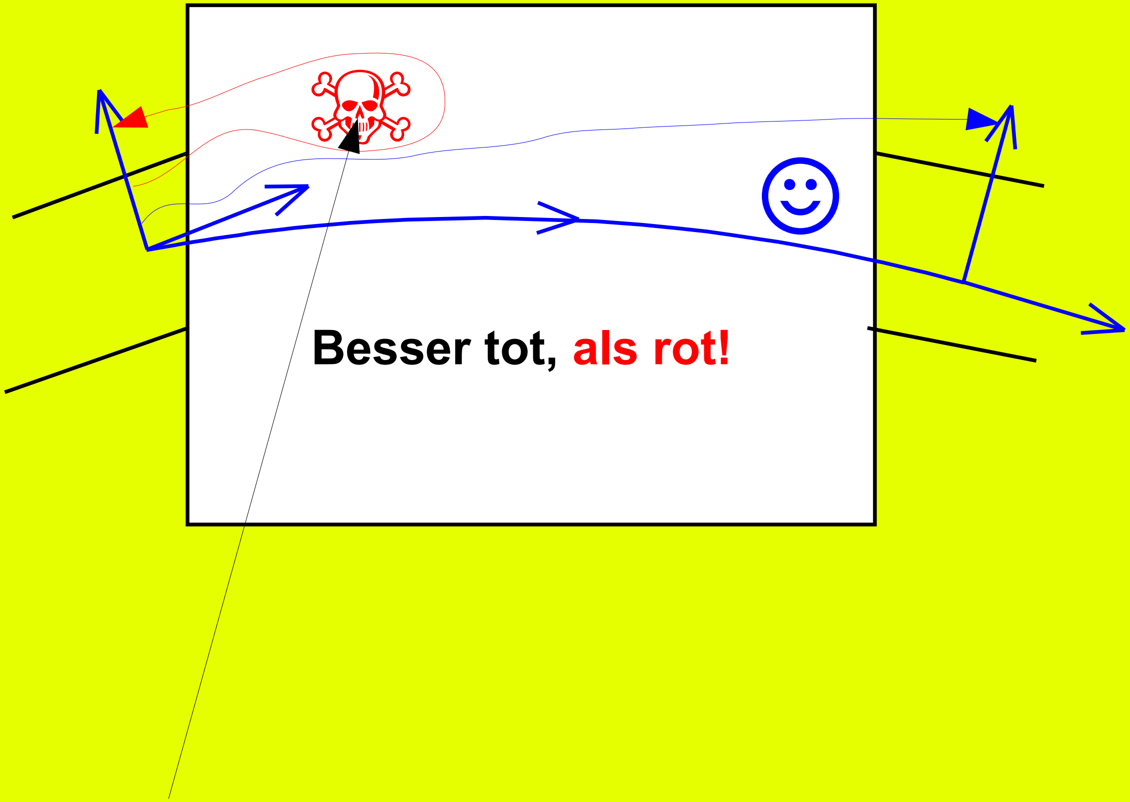
$$\mathcal{M}_{AB} = \mathcal{M}_3 \circ \mathcal{D}_2 \circ \mathcal{M}_2 \circ \mathcal{D}_1 \circ \mathcal{M}_1$$

Integration: magnet looks a Beam Line
積分：磁石はビームラインのように見える



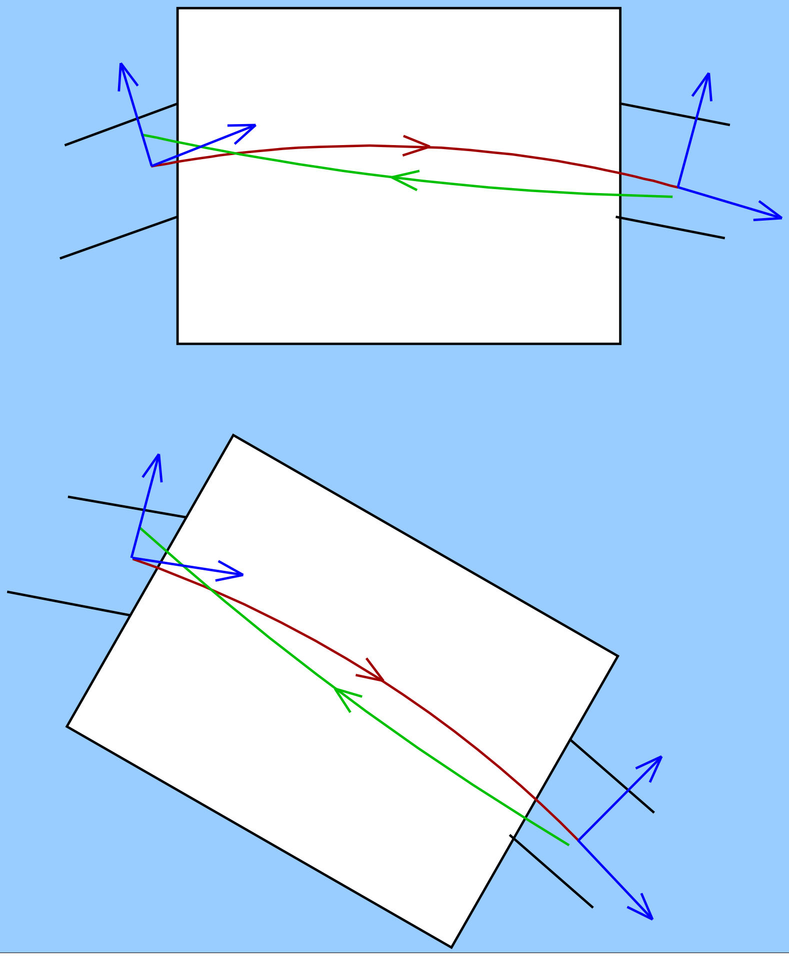
$$\mathcal{M}_2 = \mathcal{M}_{L/9} \circ \mathcal{M}_{L/9} \circ \mathcal{M}_{L/9} \circ \mathcal{M}_{L/9} \circ \mathcal{M}_{L/9} \circ \mathcal{M}_{L/9} \circ \mathcal{M}_{L/9} \circ \mathcal{M}_{L/9} \circ \mathcal{M}_{L/9}$$

他の些細な結果



Not all trajectories are considered
いくつかの軌道が計算されていない

Trajectories attached to magnets
軌道が磁石に接続されている

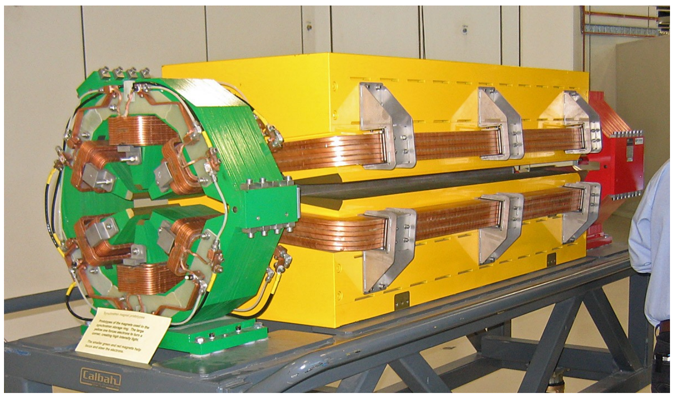


Simulation Programs ： シミュレーションのプログラム

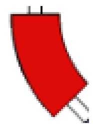
Each magnet has a unique role

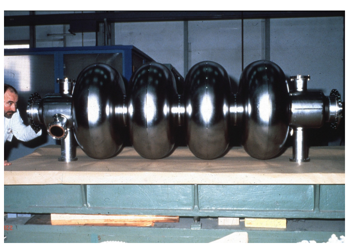
Each magnet produces its unique field

Each magnet controls the particle motion independently of other magnets

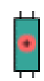


Bend

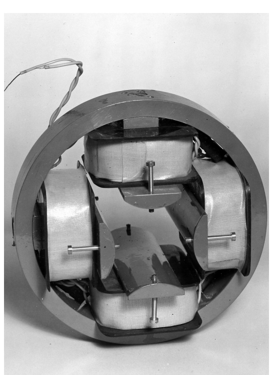


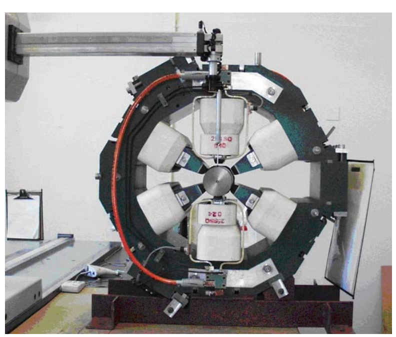


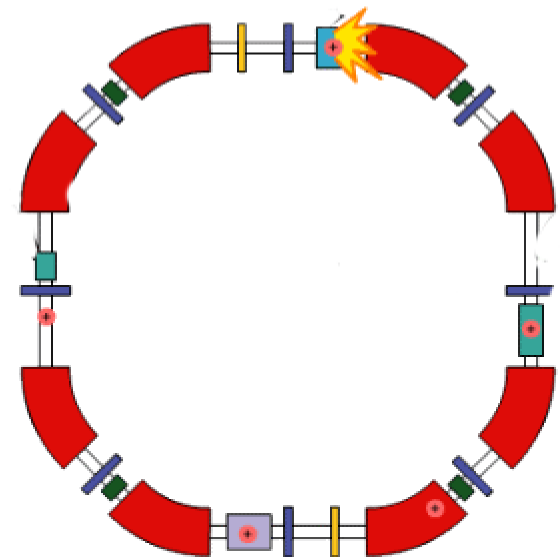
Accelerating Cavity



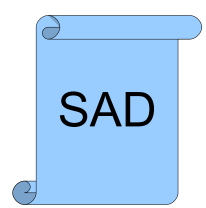
Multipoles

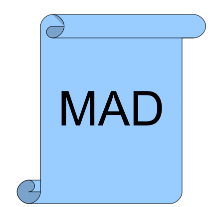






Simulation programs have a catalog of magnets
シミュレーションのプログラムは、磁石のカタログを持っている
例えば：





Summary

- Standard Laws of Physics: Detectors, Planets...

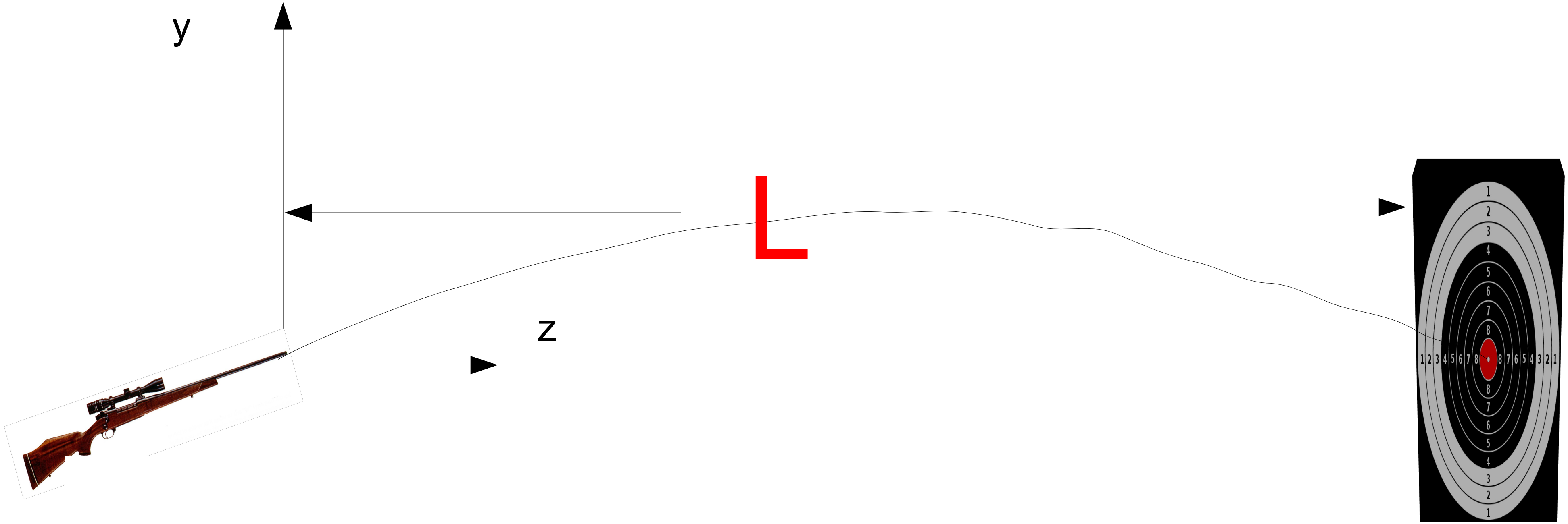
  Arrangement of hardware -> Global B -> Propagate a time dt

- Lens based approach: Accelerators, Microscopes, spectrometres,...

  hardware={set of magnets}-> each magnet -> its B field -> its propagator through it

例

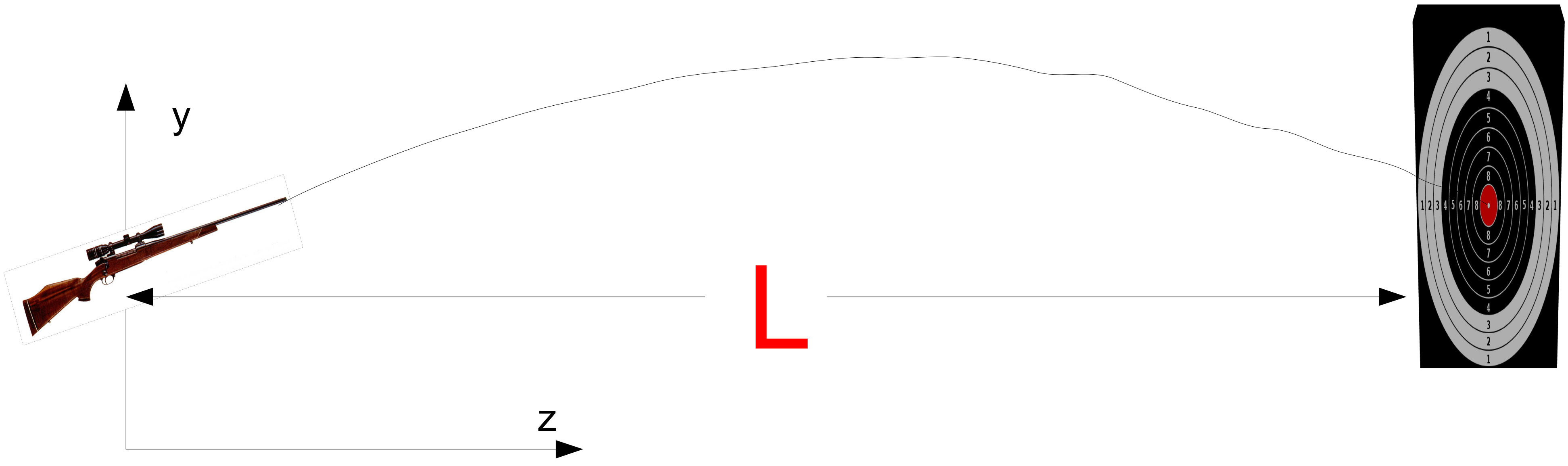
A projectile in a frictionless situation!
摩擦のない環境で発射



$$F = ma \longrightarrow \ddot{y} = -g \quad \text{and} \quad \ddot{z} = 0$$

Solution $\longrightarrow$ $$y(t) = -1/2\ g\ t^2 + v_{y0}t \qquad \text{and} \quad z(t) = v_{z0}t$$

- When we shoot, we are not interested in the trajectory as a function of time (unless you are being *counter-snipered*) but as a function of z
- Specifically $z=0$ and $z=L$!
- Starts to look like a magnet in a ring or linac!



Question

Can we use "z", the length along the trajectory to parametrize the motion?
時間の "z" の代わりに使うことは???

Answer

Yes! Provided we consider trajectories towards the target only! Better be fine with a gun!
単一方向軌道を考慮する。

If we have the time-solution...

Since we have the solution, we can invert in terms of z:

$$y(z) = -1/2\ g\ \frac{z^2}{v_{z0}^2} + \frac{v_{y0}}{v_{z0}} z \qquad \text{and} \quad t(z) = \frac{z}{v_{z0}}$$

But in general we do not have the time-solution

So we re-express the equations of motion

$$\text{in general if } \frac{d^2y}{dt^2} = F_y \text{ and } \frac{d^2z}{dt^2} = F_z$$

Chain Rule →

$$\frac{d^2y}{dt^2} = \frac{dv_y}{dt} = \frac{dv_y}{dz}v_z = F_y$$

$$\frac{dy}{dz} = v_y$$

$$\frac{dv_y}{dz} = \frac{F_y}{v_z}$$

$$\frac{dt}{dz} = \frac{1}{v_z}$$

$$\frac{dv_z}{dz} = \frac{F_z}{v_z}$$

In our gun case:

$$\frac{dy}{dz} = v_y$$

$$\frac{dv_y}{dz} = -g/v_z$$

$$\frac{dt}{dz} = 1/v_z$$

$$\frac{dv_z}{dz} = 0$$

$$\longrightarrow \begin{cases} y(z) = -1/2\ g\ \dfrac{z^2}{v_{z0}^2} + \dfrac{v_{y0}}{v_{z0}} z \\ \\ t(z) = \dfrac{z}{v_{z0}} \end{cases}$$

But physicists like the Hamiltonian!

The motion can be derived from something called the Hamiltonian which happens here to be the energy of the system!

$$H = Energy = \frac{1}{2}\left\{p_y^2 + p_z^2\right\} + gy$$ ← m=1 (more like a canon ball!)

$$\frac{d\vec{r}}{dt} = \frac{\partial H}{\partial \vec{p}} \qquad \frac{d\vec{p}}{dt} = -\frac{\partial H}{\partial \vec{r}}$$

Hamilton's Equations with Time

$$\frac{d\vec{r}}{dt} = \frac{\partial H}{\partial \vec{p}} \qquad \frac{d\vec{p}}{dt} = -\frac{\partial H}{\partial \vec{r}}$$

$$\frac{dy}{dt} = p_y$$
$$\frac{dp_y}{dt} = -g$$
$$\frac{dz}{dt} = p_z$$
$$\frac{dp_z}{dt} = 0$$

→

$$\ddot{y} = -g \quad \text{and} \quad \ddot{z} = 0$$

Same as before!

Lens version i.e. using "z"

$$H = Energy = \frac{1}{2}\left\{p_y^2 + p_z^2\right\} + gy$$

$$\downarrow$$

$$K = -p_z = -\left(2H - p_y^2 - 2gy\right)^{1/2}$$

The resulting equations are

$$\begin{aligned}
dy/dz = \partial K/\partial p_y &= p_y/\sqrt{2H - p_y^2 - 2gy} = p_y/p_z \\
dp_y/dz = -\partial K/\partial y &= -g/\sqrt{2H - p_y^2 - 2gy} = -g/p_z \\
dt/dz = -\partial K/\partial H &= 1/\sqrt{2H - p_y^2 - 2gy} = 1/p_z \\
dH/dz = \partial K/\partial t &= 0
\end{aligned}$$

Numerical results

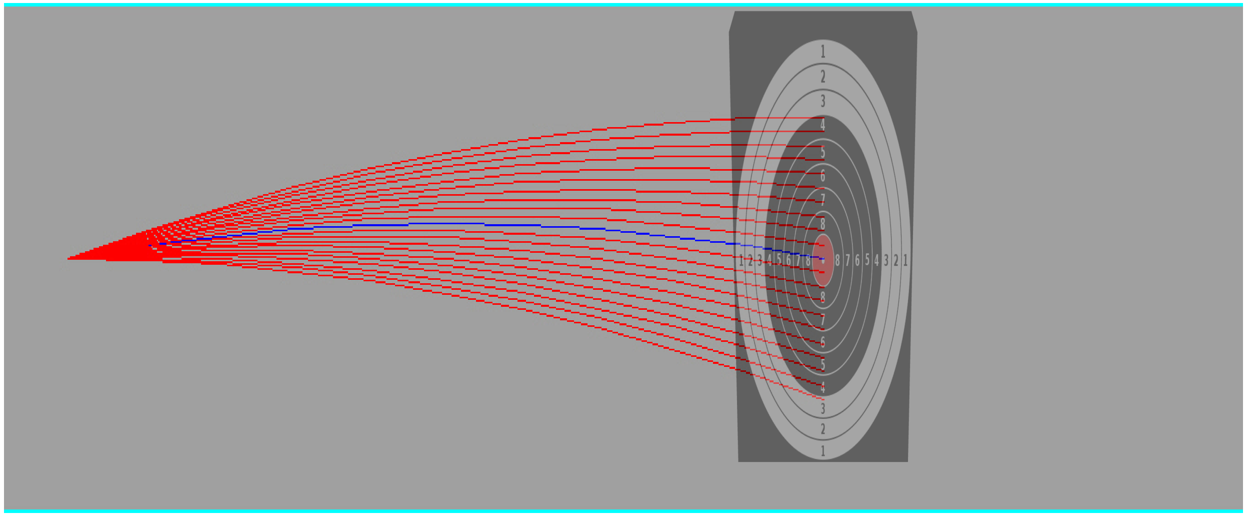
Putting some numbers into the equation

$$g = 10\ m/s^2\ ;\ v_0 = 800\ m/s\ ;\ y_0 = 0;\ L = 60\ m$$

The blue curve is some ideal trajectory.

The red curves are various shots missing the bullseye!

These were computed using the Hamiltonian K and integrating it from z=0 to z=L.



For the mathematically oriented

- The motion can be derived from minimizing a functional: ***the role of t and z seem symmetric!***

$$\delta A = \delta \int_{\vec{x}_0,t_0}^{\vec{x}_1,t_1} \vec{p}\cdot d\vec{x} - H dt \quad \longleftarrow \quad = \int L dt = \int \left(\vec{p}\cdot\dot{\vec{x}} - H\right) dt$$

$$= \int \delta\vec{p}\cdot d\vec{x} + \vec{p}\cdot d\delta\vec{x} - \delta H dt - H d\delta t$$

$$= \vec{p}\cdot\delta\vec{x} - H\delta t\,\Big|_{\vec{x}_0,t_0}^{\vec{x}_1,t_1}$$

$$+ \int \left(\delta\vec{p}\cdot d\vec{x} - d\vec{p}\cdot\delta\vec{x} - \frac{\partial H}{\partial\vec{x}}\delta\vec{x}dt - \frac{\partial H}{\partial\vec{p}}\delta\vec{p}dt - \frac{\partial H}{\partial t}\delta t dt + \delta t dH\right) dt$$

$$= \int \Big(\underbrace{\left\{ d\vec{x} - \frac{\partial H}{\partial\vec{p}} dt \right\}}_{=0} \cdot\delta\vec{p} - \underbrace{\left\{ d\vec{p} + \frac{\partial H}{\partial\vec{x}} dt \right\}}_{=0} \cdot\delta\vec{x} + \underbrace{\left\{ dH - \frac{\partial H}{\partial t} dt \right\}}_{=0} \delta t \Big) dt$$

$$
\begin{aligned}
\delta A &= \delta \int_{\vec{x}_0,t_0}^{\vec{x}_1,t_1} \vec{p}\cdot d\vec{x} - H dt = \delta \int_{\vec{x}_0,t_0}^{\vec{x}_1,t_1} \vec{p}\cdot d\vec{x} + \underbrace{p_t}_{-H} dt \\
&= \delta \int_{\vec{x}_0,t_0}^{\vec{x}_1,t_1} p_x dx + p_y dy + p_z dz + p_t dt \\
&= \delta \int_{\vec{x}_0,t_0}^{\vec{x}_1,t_1} p_x dx + p_y dy + p_t dt - \underbrace{K}_{-p_z} dz
\end{aligned}
$$

Provided z(t) is *monotonous* for the trajectories (true unless you consider pointing the rifle in your face), it seems that $K=-p_z$ could play the role of a z-parameterized Hamiltonian for the motion along the positive z-axis. (z plays the role of time)

Conclusion: we do the same for magnets

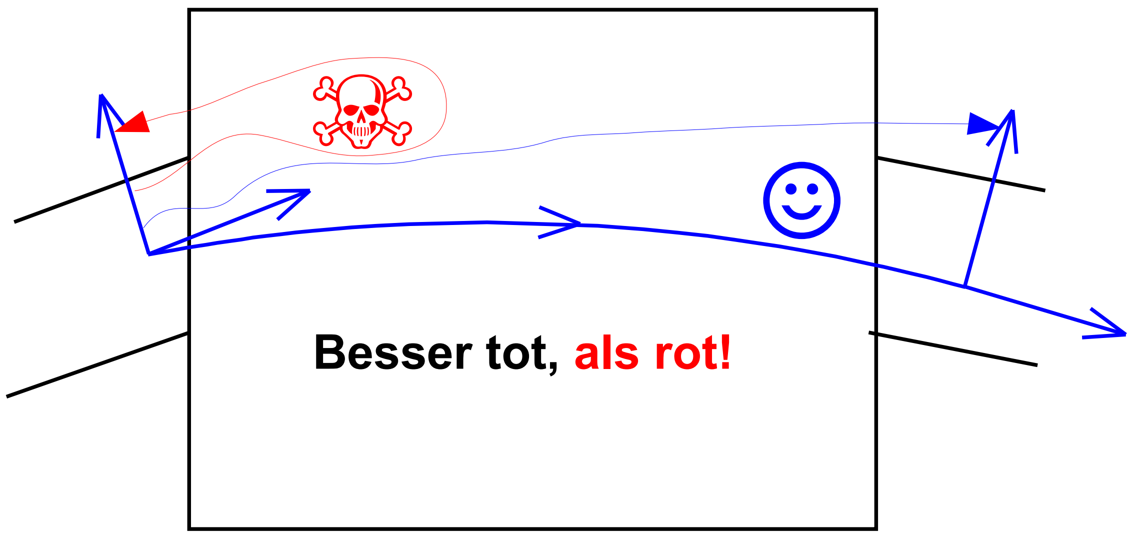
$$H = \sqrt{m^2c^4 + \left(\vec{p} - q\vec{A}\right)^2 c^2}$$

$$K = -\frac{1}{c}\sqrt{H^2 - m^2c^4 - (p_x - qA_x)^2c^2 - (p_y - qA_y)^2c^2} - qA_z$$

$-p_z$

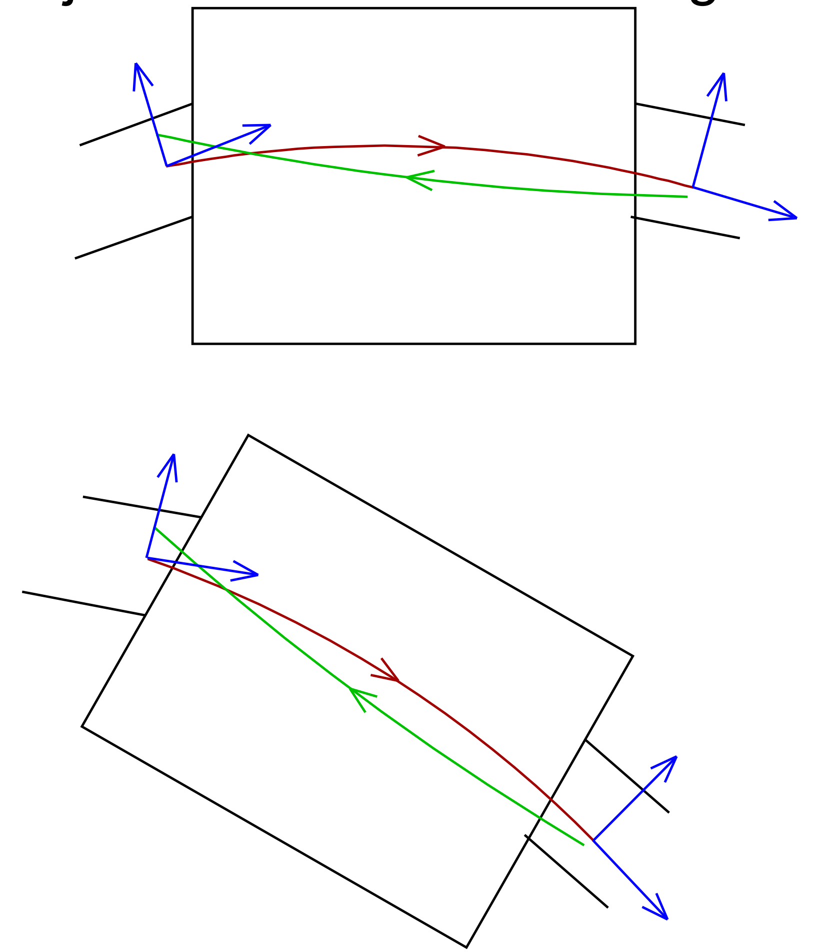
Energy and time are depedent variables

Conclusion on Lens Paradigm



Not all trajectories considered

Trajectories attached to magnets



Some simple magnets

- We will look at a quadrupole and a dipole (ideal sector bend)
- We will then write simples codes by ourself and compare with the code MAD-X of CERN
- These codes compile using the free g95 compiler.

Ideal Quadrupole

$$K = -\frac{1}{c}\sqrt{H^2 - m^2c^4 - (p_x - qA_x)^2c^2 - (p_y - qA_y)^2c^2} \; - \; qA_z$$

$$A_z \quad = \quad -\frac{B'}{2}\left(x^2 - y^2\right)$$

$$K_r = -\sqrt{1 + 2\varepsilon/\beta_0 + \varepsilon^2 - p_x^2 - p_y^2} + \frac{k}{2}\left(x^2 - y^2\right) \quad \text{where } k = \frac{B'q}{p_0}$$

K_r *is in scaled variables*

$$\varepsilon = \frac{(H - H_0)}{p_0 c} \quad \text{and} \quad p_{x,y} = \frac{p_{x,y}^{old}}{p_0}$$

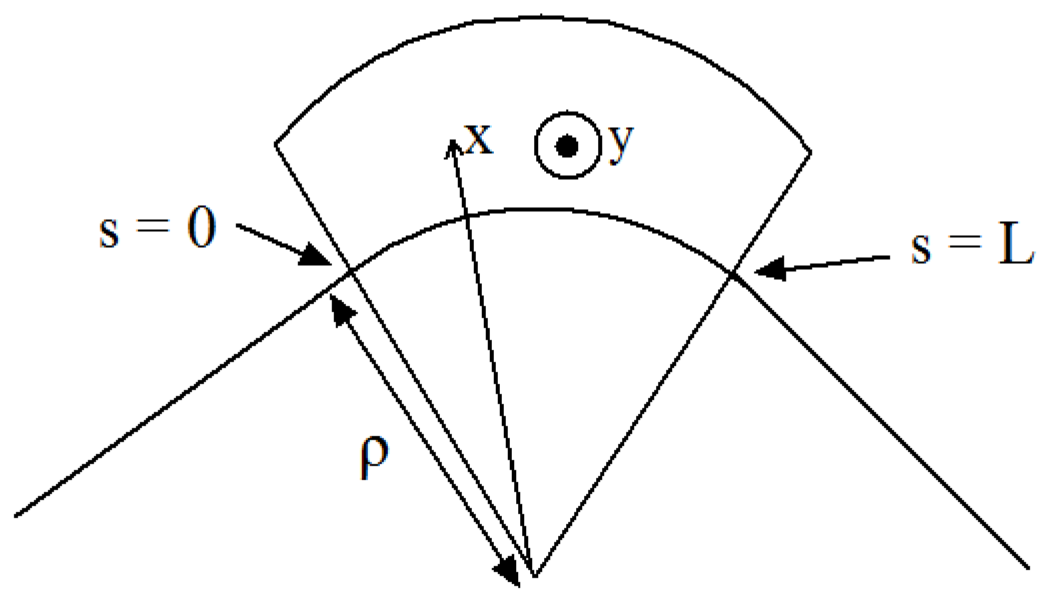
Ideal Sector Bend

$$K_r = \underbrace{-(1+hx)\sqrt{1+2\varepsilon/\beta_0+\varepsilon^2-p_x^2-p_y^2}}_{\text{Drift in Polar Coordinates}} + \underbrace{b(x+h\frac{x^2}{2})}_{\text{Multipole Kick}}$$

$$h = 1/\rho$$

$$b = h = 1/\rho$$

on design orbit



Notice the the Hamitonian are Non-linear

$$K_r = -\sqrt{1 + 2\varepsilon/\beta_0 + \varepsilon^2 - p_x^2 - p_y^2} + \frac{k}{2}\left(x^2 - y^2\right)$$

Small angles

$$\approx \frac{p_x^2 + p_y^2}{2\left(1 + 2\varepsilon/\beta_0 + \varepsilon^2\right)} + \frac{k}{2}\left(x^2 - y^2\right) + \cdots$$

Quadratic Hamiltonian: very popular in Accelerator Physics

Lead to the false belief that quadrupoles are linear elements!

$$1 + 2\varepsilon/\beta_0 + \varepsilon^2 = (1 + \delta)^2 \qquad \delta = \frac{p - p_0}{p_0} \quad \text{and} \quad \varepsilon = \frac{H - H_0}{p_0 c}$$

Sector Bend

$$K_r = -(1+hx)\sqrt{1+2\varepsilon/\beta_0+\varepsilon^2-p_x^2-p_y^2}+b\left(x+h\frac{x^2}{2}\right)$$

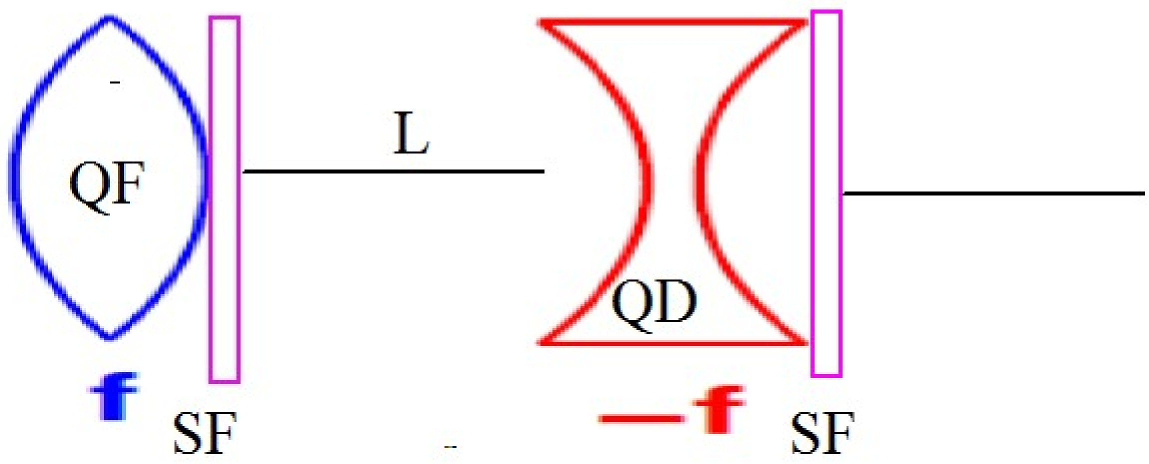
Small angles

$$\approx \frac{p_x^2+p_y^2}{2\left(1+2\varepsilon/\beta_0+\varepsilon^2\right)}-hx\delta+b\left(x+h\frac{x^2}{2}\right)-hx\cdots$$

If b=h => ideal orbit. No linear terms in the Hamiltonian

Again, sector bends appear linear!

System for which we will write a small code: next lecture



$$K_q = -p_t + \frac{p_x^2 + p_y^2}{2(1+p_t)} + k_Q \frac{x^2 - y^2}{2}$$

$$K_s = -p_t + \frac{p_x^2 + p_y^2}{2(1+p_t)} + Lk_S\delta(s - s_0)\frac{x^3 - 3xy^2}{3}$$

Solution for thin sextupoles

$$p_x^{\mathrm{final}} = p_x - Lk_S(x^2 - y^2) \qquad \text{and} \qquad p_y^{\mathrm{final}} = p_y + Lk_S 2xy$$