# Reinventing the Wheels

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*So, no, you shouldn't reinvent the wheel.* ***Unless you plan on learning more about wheels****, that is.*

https://blog.codinghorror.com/dont-reinvent-the-wheel-unless-you-plan-on-learning-more-about-wheels/

# A kinematical method to evaluate synchrotron radiation

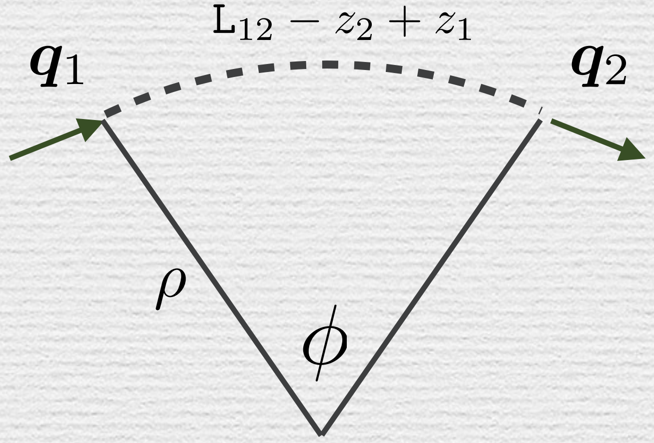
Let $\boldsymbol{q}$ denote the orientation vector of the momentum of a particle:

$$\boldsymbol{q} = \left(\frac{p_x}{p}, \frac{p_y}{p}, \frac{p_z}{p}\right)$$
$$p_z = \sqrt{p^2 - p_x^2 - p_y^2}\ .$$

Suppose a particles traverses a section (1, 2) of an accelerator component, then the orientation changes from $\boldsymbol{q}_1$ to $\boldsymbol{q}_2$. The bending angle $\phi$ and the radius of curvature $\rho$ are approximated, assuming a uniform bending, by:

$$\sin|\phi| = |\boldsymbol{q}_2 \times \boldsymbol{q}_1|$$
$$\rho = \frac{\mathtt{L}_{12} - z_2 + z_1}{|\phi|},$$

where $\mathtt{L}_{12}$ is the nominal length of the component between 1 and 2, and $z_{1,2}$ are the values of longitudinal coordinate $z \equiv -v(t - t_0)$ at the locations 1 and 2.



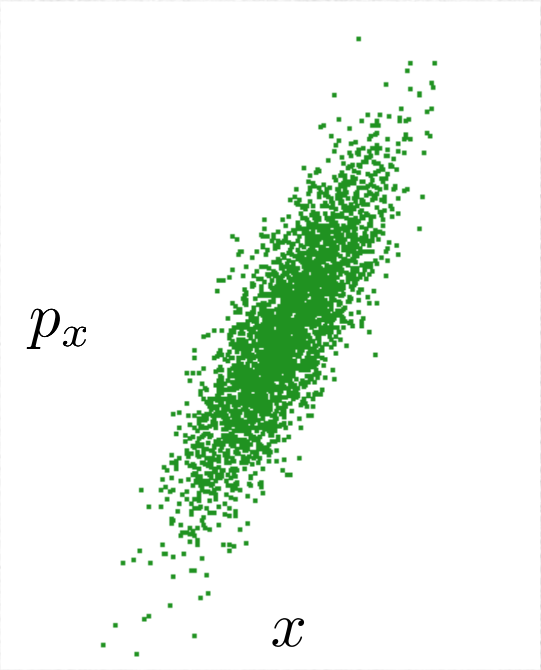
# Merits of the kinematical method

By knowing $\phi$ and $\rho$ as well as the momentum of the particle, we can derive all information about the emission of synchrotron radiation (if we can use a classical formula with uniform bending).

- Thus the synchrotron radiation can be handled *by a single routine for any type of component*, such as multipole, solenoid, fringe field, even including electric field, without knowing details of the field.
- Not only the radiation itself, its derivatives by phase space coordinated can be obtained kinematically using the transfer matrix.
- This method may be applied for a *spin motion* if the longitudinal filed is taken care properly.

# Emittance by Gaussian Fit

Q: How can we evaluate the emittance $\varepsilon_x$ and Twiss parameters $\alpha_x$, $\beta_x$ af particles in the phase space $(x, p_x)$ using Gaussian fit?

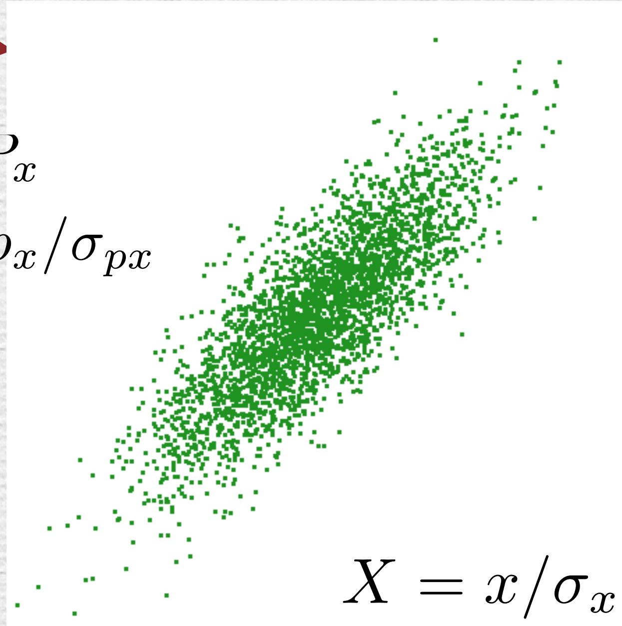



A: A possible way, assuming the center of mass is $(0, 0)$:

- First, obtain $\sigma_x$ and $\sigma_{px}$ by 1D Gaussian fits, respectively.
- Then normalize $x$ and $p_x$ by $\sigma_x$ and $\sigma_{px}$, respectively:

$$X \equiv \frac{x}{\sigma_x} ,$$
$$P_x \equiv \frac{p_x}{\sigma_{px}} .$$

- We notice in the phase space $(X, P_x)$, the beam is *always 45° tilted!*




- Thus the emittance in the $(X, P_x)$ space is the product of sigmas $\sigma_\pm$ of:

$$u_\pm \equiv \frac{X \pm P_x}{\sqrt{2}} ,$$

  which are again obtained by 1D Gaussian fits.

- The emittance in the original $(x, p_x)$ space, $\alpha$, and $\beta$ are written as:

$$\varepsilon_x = \sigma_x \sigma_{px} \sigma_+ \sigma_-$$
$$\alpha_x = r - \frac{1}{r} , \qquad \beta_x = \frac{\sigma_x^2}{\varepsilon_x} ,$$

  where $r \equiv \sigma_- / \sigma_+$.

- Above has been implemented as `FitEmit[x, px]` in SAD.

# Confidential interval of functions of fit parameters

Let us consider a multidimensional fit of data:

$$(x_1, y_1), ..., (x_n, y_n)$$

by a function $f(x; a_1, ..., a_m)$ with fit parameters $\boldsymbol{a} = (a_1, ..., a_m)$. This is done by minimizing

$$\chi^2 = \sum_{i=1}^{n} \frac{(y_i - f(x_i; \boldsymbol{a}))^2}{\Delta y_i^2} \ ,$$

where $\Delta y_i$ is the standard deviation of the data at $x_i$.

At the optimum point of $\boldsymbol{a}$, The confidence interval of parameters $a_i$ corresponding to 68% confidence is expressed as:

$$\Delta a_i = \pm\sqrt{\Delta\chi_m^2}\sqrt{C_{ii}} \ ,$$

where $C_{ii}$ is the $i$-th diagonal part of the *covariance matrix* C, and $\Delta\chi_m/2$ is the solution of $Q(m/2, x) = \mathrm{erfc}(1/\sqrt{2})$. The matrix C is written as

$$C_{jk} = \sum_{i=1}^{m} \frac{1}{w_i^2} V_{ji} V_{ki} \ ,$$

where the matrix V and singular value $w_i$ are the singular decomposition (SVD) of the *design matrix*

$$\begin{aligned} \mathrm{A} = (A_{ik}) &= \frac{1}{\Delta y_i} \frac{\partial f(x_i; \boldsymbol{a})}{\partial a_k} \\ &= \mathrm{U}^T \mathrm{diag}(w_1, ..., w_m) \mathrm{V} \ . \end{aligned}$$



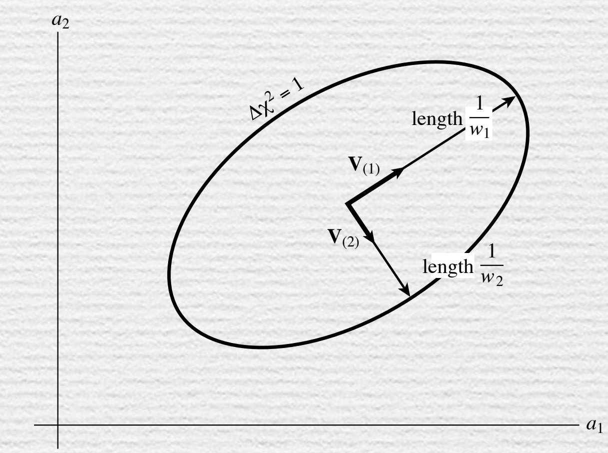



Figure 15.6.5. Relation of the confidence region ellipse $\Delta\chi^2 = 1$ to quantities computed by singular value decomposition. The vectors $\mathbf{V}_{(i)}$ are unit vectors along the principal axes of the confidence region. The semi-axes have lengths equal to the reciprocal of the singular values $w_i$. If the axes are all scaled by some constant factor $\alpha$, $\Delta\chi^2$ is scaled by the factor $\alpha^2$.



cf. Numerical Recipes for FORTRAN 77 $$15.6

# Confidential interval of functions of fit parameters (cont'd)

Then the question is how to estimate the confidence interval of other set of parameters $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{a})$. Actually (I think) it is not possible to calculate them only with the intervals for $\boldsymbol{a}$ and the relation $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{a})$ in general.

The design matrix for parameters $\boldsymbol{b}$ should become:

$$\mathrm{B} = (B_{ik}) = \frac{1}{\Delta y_i} \frac{\partial f(x_i; \boldsymbol{a})}{\partial a_j} \frac{\partial a_j}{\partial b_k} = \mathrm{AM} \ ,$$

$$\mathrm{M} = \left( \frac{\partial a_j}{\partial b_k} \right) \ .$$

To define the covariance matrix associated with B, we need the SVD of B, which cannot be obtained by the SVD of A without a full calculation of the SVD of B, unless M has special characteristics.

$$\mathrm{A} = \mathrm{U}^T \mathrm{diag}(w_1, ..., w_m) \mathrm{V} \ ,$$

$$\mathrm{B} = \mathrm{AM} = \mathrm{U}'^T \mathrm{diag}(w'_1, ..., w'_m) \mathrm{V}' \ .$$

The `Fit` or `FitPlot` functions in SAD have not output the design matrix A, which is necessary to calculate the SVD of B or the confidence interval of $\boldsymbol{b}$. Now they return it under the tag `DesignMatrix`.

# Synchrotron radiation in a transport line

A formula for transverse emittance generated by synchrotron radiation something like

$$\varepsilon_x \propto \int \mathcal{H}_x/|\rho|^3 ds \ ,$$

with

$$\mathcal{H}_x \equiv \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_x^2$$

is not applicable to a single-pass beam transport line, since the above assumes a matching to the ring Twiss parameters after many turns of circulation.

For an extreme case, if the radiation source is very thin, the radiation-excited particles will align on a straight line $(R_{16}\delta, R_{26}\delta)$ in the phase space, and its emittance is zero. Due to chromaticity, etc., the line will be smeared to have some finite emittance, but it is not expressed by the above expression.